MAC-CPTM Situations Project

Situation 02: Parametric Drawings

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Prompt

This example, appearing in CAS-Intensive Mathematics (Heid and Zbiek, 2004)[1], was inspired by a student mistakenly grabbing points representing both parameters (A and B in f(x) = Ax + B) and dragging them simultaneously (the difference in value between A and B stays constant). This generated a family of functions that coincided in one point. Interestingly, no matter how far apart A and B were initially, if grabbed and moved together, the graphs of the functions in the family always coincided on the line x = -1.

Commentary

In this case, GSP was a vehicle that brought mathematical relationships to the fore. When one encounters such a phenomenon, one can enhance the experience by noticing the potential for mathematics in the patterns that are seen. Focus 1 uses transformations to explain the graphical phenomenon, while focus 2 uses a symbolic proof. Focus 3 extends the phenomenon to quadratic functions (which also appears in CAS-Intensive Mathematics). In addition, focus 3 considers polynomials of higher degree (which generated another interesting relationship along with its proof). Focus 3 illustrates the decisions needed in designing an extension of a mathematical generalization.

Mathematical Foci

Mathematical Focus 1

Graphical phenomenon can be explained in terms of transformations.

Let $y = \mathrm{A}x + \mathrm{B}$,
Suppose A – B = k, a constant.
If A – B = k, then we also know that B = A - k.

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[1] Heid, M. K. & Zbiek, R. M. (2004). The CAS-Intensive Mathematics Project. NSF Grant No. TPE 96-18029

Using substitution, we can rewrite y=Ax + B as y = Ax + (A – k), and equivalently, y = A(x+1) – k

The graphs of the family of lines given by y = A(x+1) – k result from translating and rotating the graph of $y = x$.

First consider the family of lines given by y = (x+1) – k. When the value of k, constant for each family of lines, is changed, it shifts the family of lines given by y=(x+1) up or down vertically along the line x=-1.

Now consider the family of lines given by y = A(x+1) - k. Since A represents the slope of any line in the family, when A is changed, it causes the rotation of the graph y=A(x+1) - k about the point (-1, -k). Thus, the lines of the family y = A(x+1) - k will intersect in the point (-1, -k).

Figure 1 displays a screen dump after the values of A and B have been simultaneously dragged.

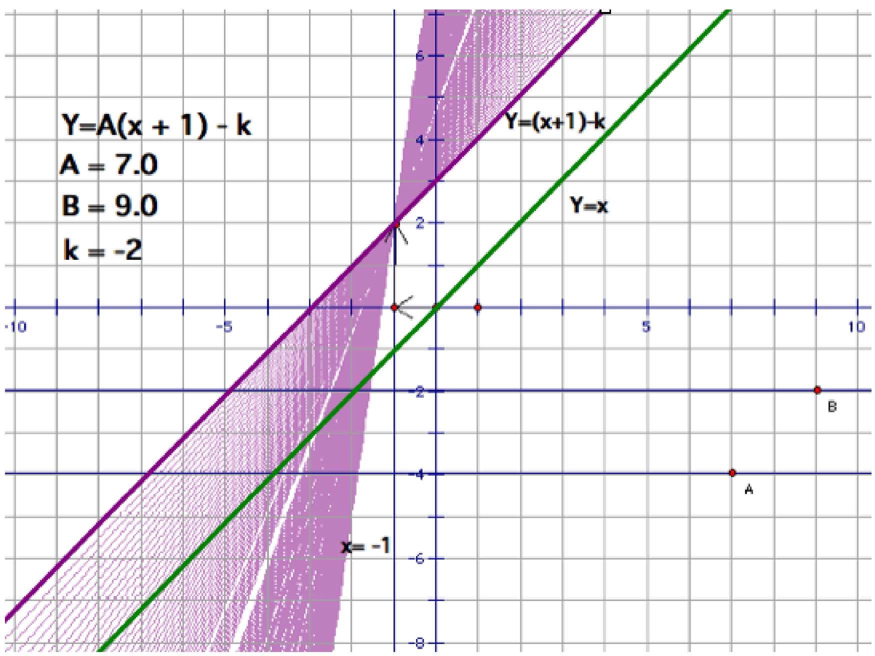


Figure 1 Screen dump showing trace of $f(x) = Ax + B$ after A and B have been dragged simultaneously.

Mathematical Focus 2

Graphical phenomenon can frequently be explained using symbolic proof.

Figure 2 displays a screen dump after A and B have been simultaneously dragged.

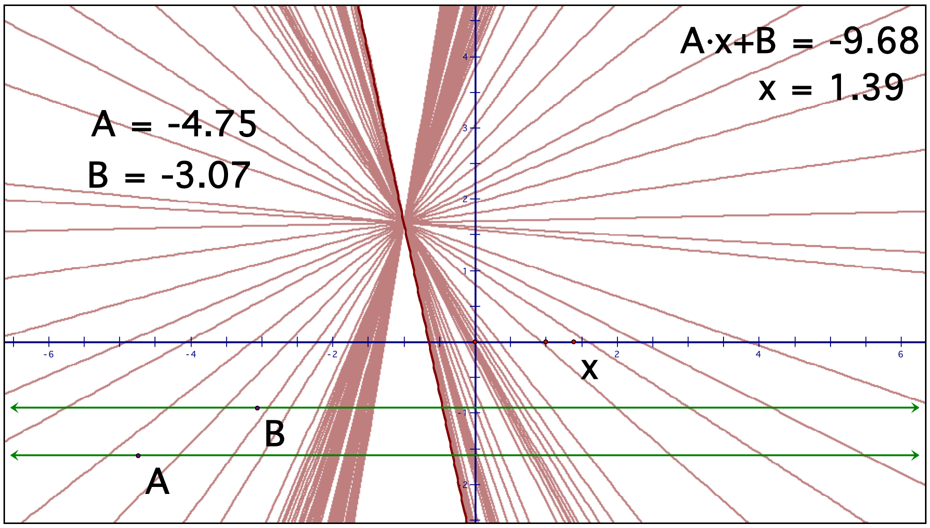


Figure 2. Screen dump showing trace of $f(x) = Ax + B$ after A and B have been dragged simultaneously.

Notice that the family of lines that appear intersect on the line $x = -1$. To explain why this will be the case for any value of A and B where their difference remains constant can be explained using the following symbolic proof:

Let $y = Ax + B$ and suppose A – B = k, a constant. Let $y_1 = A_1x + B_1$ and $y_2 = A_2x + B_2$ be two lines in this family. $A_1 - B_1 = k = A_2 - B_2 \Rightarrow A_1 - A_2 = B_1 - B_2$. To determine the point of intersection, y_1 is set equal to y_2. Therefore,

$$A_1x + B_1 = A_2x + B_2$$
$$(A_1 - A_2)x = B_2 - B_1$$
$$x = \frac{B_2 - B_1}{A_1 - A_2} = \frac{-(B_1 - B_2)}{A_1 - A_2} = -1$$

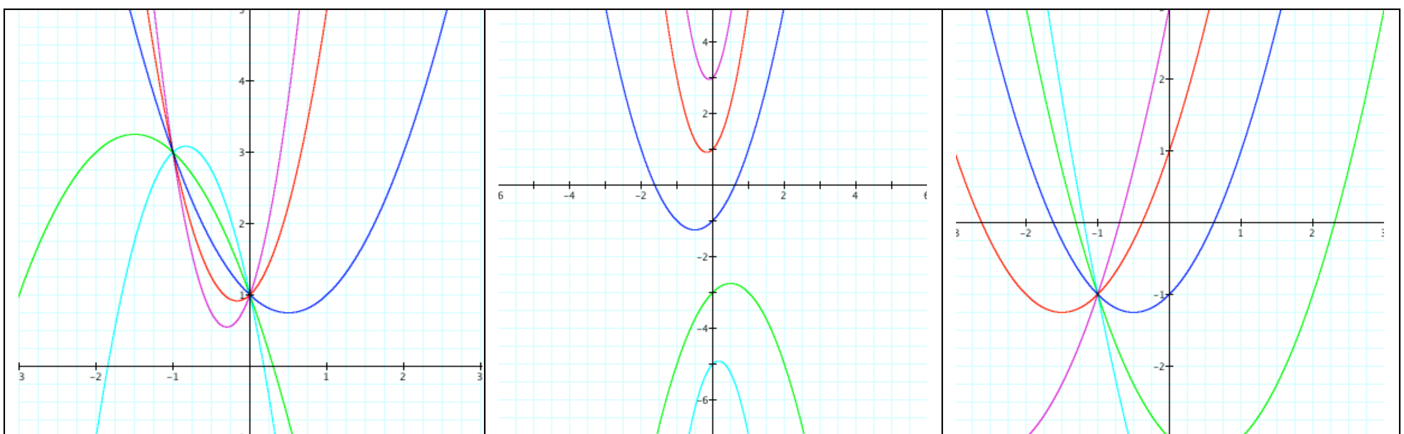
Hence, any two lines in this family will intersect at the point (-1, -k).

Mathematical Focus 3

Many times phenomena that are observed for certain functions can be extended to other functions with similar properties. For example phenomena related to linear functions extended to polynomial functions with degree > 1.

When looking at quadratic functions, additional assumptions must be made to investigate the phenomenon. In a quadratic function, there are three coefficients (A, B, and C in $y = Ax^2 + Bx + C$) rather than two (A and B in $y = Ax + B$). So, for the quadratic function, $y = Ax^2 + Bx + C$, we will consider three possibilities in investigating an extension of the phenomenon we observed in the linear function: Case 1: The difference between A and B is constant. We would hold C constant with A – B = k; Case 2: The difference between A and C is constant. We would hold B constant with A – C = k; Case 3: The difference between B and C is constant. We would hold A constant with B – C = k. In each of these cases, symbolic proofs similar to the one in Focus 1 can be developed, and the following conclusions can be drawn:

1. If C is held constant, there will be two intersections at (0, C) and (-1, k + C).
2. If B is held constant, there will be no intersection because the system of equations is equivalent to $x^2 = -1$.
3. If A is held constant, there will be one intersection at (-1, A – k).



C = 1, A – B = 2 | B = 1, A – C = 2 | A = 1, B – C = 2

This idea can now be extended for polynomials of the n^th degree. In order to do this extension, all but two coefficients are held constant and the remaining two have a constant difference. Consider the polynomial $y = A_n x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + \ldots + A_2x^2 + A_1x + A_0$. Choose 2 coefficients to vary, but keep their difference constant. All other coefficients will be held constant. Suppose $A_j - A_i = k,\ i < j$. If we have two polynomials in this family, we can determine where they will intersect by setting them equal to each other. Using a symbolic proof we see that the intersections will occur at the following points:

$$x^i = 0, \quad y = A_0$$
$$x^{j-i} = -1, \quad y = A_n(-1)^n + A_{n-1}(-1)^{n-1} + \ldots + A_0$$

However, depending on the values of i and j, these points may or may not be defined. Specifically, $x^{j-i} = -1$ will be defined as a real number only where $j - i$ is an odd number.